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Comment on the Representation of Splines as Boolean Sums

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1. INTRODUCTION

In the past few years, Boolean sums of interpolating projectors have figured prominently in the formulation of blended interpolation, pioneered by, for example, Gordon and Hall [3, 4], and in the representation of some finite elements. More recently, Boolean sums have been used to obtain other types of interpolants, including splines.

Generally, if $P: X \to \operatorname{Im} P$ and $Q: X \to \operatorname{Im} Q$ are projectors, with $\operatorname{Im} Q \subset X$, then the *Boolean sum* $P \oplus Q$ is defined by $(P \oplus Q)f = P(f - Qf) + Qf$, $\forall f \in X$, and $(P \oplus Q)f \in \operatorname{Im} P \oplus \operatorname{Im} Q$. Further, if $\operatorname{Im} P \subset X$ and $QP = Q$, then $P \oplus Q$ is again a projector. In our applications X is usually a function space, $\operatorname{Im} P \cap \operatorname{Im} Q = \emptyset$, both $\operatorname{Im} P$ and $\operatorname{Im} Q$ being subsets, or more frequently subspaces, of X.

In the blending context, take $X = C\{[a, b] \times [c, d]\}$. The intervals are partitioned so that $a = x_1 < x_2 < \cdots < x_m = b$, $c = y_1 < y_2 < \cdots < y_n = d$. Let P_x and P_y be interpolating projectors on X in the sense that, for $f \in X$,

$$(P_x f)(x_i, y) = f(x_i, y), \qquad (P_y f)(x, y_j) = f(x, y_j), \quad i = 1, ..., m; j = 1, ..., n.$$

It is then found that $P_x \oplus P_y$ is a projector with the "transfinite" interpolation property

$$(P_x \oplus P_y) f|_{x = x_i} = f(x_i, \cdot), \qquad (P_x \oplus P_y) f|_{y = y_j} = f(\cdot, y_j),$$
$$i = 1, ..., m; \; j = 1, ..., n.$$

Another application of Boolean sums occurs in the improvement of the approximating properties of interpolating projectors. For example, one may have an interpolating projector P on a function space X containing

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$\mathscr{P}_1$, such that $\operatorname{Im} P$ does not contain $\mathscr{P}_1$. If $Q: X \to \mathscr{P}_1$ is a projector and $QP = Q$, then $P \oplus Q$ is an interpolating projector with the additional property $(P \oplus Q)f = f$, $\forall f \in \mathscr{P}_1$.

Boolean sums also occur implicitly in kriging, a term used by geostatisticians for a process of estimating the value of a function (of two variables) at a point x_0, in terms of known values at other points x_i. In one of its formulations it can be shown that kriging involves an interpolating projector P whose image is the linear span of functions φ_i which are translates to the x_i of a basic function φ. The latter is a generalized covariance which happens to also be a function. Quite often, $\varphi(x, y) = \{x^2 + y^2\}^{1/2}$; $\varphi(x, y) = \{x^2 + y^2\}^{3/2}$ is also used. The projector Q is of weighted least-squares type, but with an indefinite weight matrix, and $\operatorname{Im} Q = \mathscr{P}_m$, where m is typically 0 or 1.

Surface splines, as developed, for example, by Duchon [2] and Meinguet [6], can be expressed as Boolean sums much as in the kriging method. Some preliminary studies of this connection have been announced by Salkauskas [8]; a more thorough and profound account of the formal equivalence of kriging and surface splines is given by Matheron [5].

While the above are most often applied in the bivariate case, there are interesting formulations of univariate splines as Boolean sums. For example, most C^2 piecewise cubic spline interpolants of $f \in X$ can be represented as $(P \oplus Q)f$, where $P: X \to \operatorname{span}\{|x - x_i|^3\}_{i=1}^n$ is an interpolating projector, and Q is a certain projector onto $\mathscr{P}_1$. In the sequel we discuss properties of the above, as well as generalizations to higher-degree univariate splines. Since our projectors always sample $f \in X$ at n distinct points x_i, it will be sufficient to assume that $X = \mathbb{R}^n$. Also, it is clear that now $QP\mathbf{f} = Q\mathbf{f}$, for Q "sees" the same values in $P\mathbf{f}$ as in $\mathbf{f}$. Consequently $P \oplus Q$ will always be a projector.

2. Univariate Splines as Boolean Sums

Suppose that $x_1 < x_2 < \cdots < x_n$ are given. In [1] it is shown that the matrices

$$V = [|x_i - x_j|^{2k+1}]_{1 \leqslant i, j \leqslant n}$$

are non-singular. We will assume that $n > 2k - 1$. Hence for any $\mathbf{f} \in \mathbb{R}^n$ there is a unique $\mathbf{c} \in \mathbb{R}^n$ such that $s(x) = \sum_{j=1}^n c_j |x - x_j|^{2k+1}$ interpolates the given data, i.e., $s(x_i) = f_i$, $i = 1, \ldots, n$. The corresponding interpolating projector may be represented as

$$P\mathbf{f} = [|x - x_1|^{2k+1}, |x - x_2|^{2k+1}, \ldots, |x - x_n|^{2k+1}]\, V^{-1}\mathbf{f}. \qquad (2.0)$$

We see that $P\mathbf{f}$ is piecewise polynomial of degree $2k+1$ with knots $x_1, ..., x_n$ and of continuity class C^{2k}.

Now let $B = [x_i^{j-1}]_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant 2k}$ and for $W \in \mathbb{R}^{n \times n}$ positive definite on the image of B, let Q_W be the corresponding weighted least-squares projector defined by

$$Q_W \mathbf{f} = [1, x, ..., x^{2k-1}](B^t W B)^{-1} B^t W \mathbf{f} \in \mathscr{P}_{2k-1}.$$

THEOREM 2.1. *Let $s(x)$ be a spline of degree $(2k+1)$ with knots $x_1 < x_2 < \cdots < x_n$ and $\mathbf{s} \in \mathbb{R}^n$ be the vector of function values of $s(x)$ at the knots. If $\mathbf{s} \notin \operatorname{Im} B$,* i.e., $\mathbf{s}$ *is* not *polynomial of degree $(2k-1)$, then there exists a W, positive definite on* Im B, *such that*

$$s(x) = (P \oplus Q_W)\mathbf{s}. \tag{2.1}$$

Further, if $\mathbf{s} \in \operatorname{Im} B$ then there exists a W for which (2.1) *holds if and only if $s(x)$ is itself a polynomial of degree $2k-1$.*

Proof. As $s(x)$ is a spline of degree $(2k+1)$, $s^{(2k)}(x)$ must be piecewise linear with knots $x_1 < x_2 < \cdots < x_n$. Hence there are coefficients α_i such that

$$s^{(2k)}(x) = \sum_{i=1}^{n} \alpha_i \, |x - x_i|.$$

Upon integrating $2k$ times, we see that

$$s(x) = \sum_{j=1}^{n} c_j \, |x - x_j|^{2k+1} + \sum_{j=0}^{2k-1} a_j x^j \tag{2.2}$$

for some coefficients c_j and a_j. Therefore,

$$\mathbf{s} = V\mathbf{c} + B\mathbf{a}. \tag{2.3}$$

Now, explicitly,

$$\begin{aligned} P \oplus Q_W \mathbf{s} = {} & [|x - x_1|^{2k+1}, ..., |x - x_n|^{2k+1}] \, V^{-1}(I - BM)\mathbf{s} \\ & + [1, x, ..., x^{2k-1}] \, M\mathbf{s}, \end{aligned} \tag{2.4}$$

where

$$M=(B'WB)^{-1}B'W.$$

Comparing (2.2) with (2.4) we see that $s=P\oplus Q_W\mathbf{s}$ if

$$V^{-1}(I-BM)\mathbf{s}=\mathbf{c} \qquad \text{and} \qquad BM\mathbf{s}=B\mathbf{a},$$

or equivalently,

$$(I-BM)\mathbf{s}=V\mathbf{c} \qquad \text{and} \qquad BM\mathbf{s}=B\mathbf{a}.$$

But by (2.3), $V\mathbf{c}=\mathbf{s}-B\mathbf{a}$ and so $s=P\oplus Q_W\mathbf{s}$ if

$$(I-BM)\mathbf{s}=\mathbf{s}-B\mathbf{a} \qquad \text{and} \qquad BM\mathbf{s}=B\mathbf{a},$$

which reduces to the single condition

$$BM\mathbf{s}=B\mathbf{a},$$

which itself, as B is of full rank, reduces to the condition

$$M\mathbf{s}=\mathbf{a}.$$

If $\mathbf{s}\in \operatorname{Im} B$, that is, $\mathbf{s}=B\mathbf{b}$, then

$$M\mathbf{s}=(B'WB)^{-1}B'WB\mathbf{b}=\mathbf{b},$$

and

$$(I-BM)\mathbf{s}=B\mathbf{b}-B\mathbf{b}=\mathbf{0},$$

so that by (2.4), $s=P\oplus Q_W\mathbf{s}$ only if $s(x)=[1, x, ..., x^{2k-1}]\,M\mathbf{s}$, i.e., $s(x)$ is a polynomial of degree $(2k-1)$. Clearly, in this case any positive definite W provides $s=P\oplus Q_W\mathbf{s}$.

If $\mathbf{s}\notin \operatorname{Im} B$, we must show that a W may be found such that $M\mathbf{s}=\mathbf{a}$. But there are certainly many matrices, $A\in\mathbb{R}^{2k\times n}$ (recall that we assume that $n>2k-1$), with the property that

$$AB=I_{2k} \qquad \text{and} \qquad A\mathbf{s}=\mathbf{a}.$$

Let $W = A'A$. If $\mathbf{x} = B\mathbf{b} \in \operatorname{Im} B$ then $\mathbf{x}'W\mathbf{x} = \mathbf{b}'B'A'AB\mathbf{b} = \mathbf{b}'I_{2k}'I_{2k}\mathbf{b} = \mathbf{b}'\mathbf{b}$ and W is positive definite on Im B. Further,

$$\begin{aligned}(B'WB)^{-1} B'W\mathbf{s} &= (B'A'AB)^{-1} B'A'A\mathbf{s} \\ &= ((AB)'\,AB)^{-1}\,(AB)'\,\mathbf{a} \\ &= I_{2k}^{-1}I_{2k}'\mathbf{a} \\ &= \mathbf{a}.\end{aligned}$$

The result follows. ∎

In the case of natural splines there is a somewhat more specific result concerning the weight matrix W of the projector Q, preliminary to which we require the following two lemmas. Let $C = [x_i^{j-1}]_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k+1}$. C resembles B (in fact they are equal when $k = 1$) but includes monomials up to degree k only.

LEMMA 2.2. $(-1)^{k-1}\, V$ *is positive definite on the kernel of* C'.

Remark. This is a special case of the results of Micchelli [7]. We would, however, offer the following completely elementary proof.

Proof. Let $\mathbf{0} \neq \mathbf{c} \in N(C')$ and $s(x) = \sum_{i=1}^{n} c_i\, |x - x_i|^{2k+1}$. Now an easy calculation shows that for $k+1 \leqslant j \leqslant 2k$, $s^{(j)}(x_1) = s^{(j)}(x_n) = 0$, that is, $s(x)$ is a *natural* spline. We now integrate $\int_{x_1}^{x_n} (s^{(k+1)}(x))^2\, dx$, by parts, k times. By the natural spline end conditions we have

$$\begin{aligned}&\int_{x_1}^{x_n} s^{(k+1)}(x)\, s^{(k+1)}(x)\, dx \\ &= (-1)^k \sum_{j=1}^{n-1} \int_{x_j}^{x_{j+1}} s^{(1)}(x)\, s^{(2k+1)}(x)\, dx \\ &= (-1)^k \sum_{j=1}^{n-1} s^{(2k+1)}\,|_{(x_j, x_{j+1})} \int_{x_j}^{x_{j+1}} s^{(1)}(x)\, dx \\ &= (-1)^k \sum_{j=1}^{n-1} s^{(2k+1)}\,|_{(x_j, x_{j+1})} (s(x_{j+1}) - s(x_j)).\end{aligned}$$

A manipulation of the sum gives, for the above,

$$(-1)^{k-1} \sum_{j=1}^{n} s(x_j)\{\text{jump in } s^{(2k+1)} \text{ at } x_j\}.$$

But an elementary calculation reveals that $\{\text{jump in } s^{(2k+1)} \text{ at } x_j\} = 2(2k+1)!\, c_j$ and we see that

$$2(-1)^{k-1}(2k+1)! \sum_{j=1}^{n} c_j s(x_j) = \int_{x_1}^{x_n} (s^{(k+1)}(x))^2\, dx.$$

As $\mathbf{c} \neq \mathbf{0}$, $s(x)$ is certainly not a polynomial of degree k and hence

$$\int_{x_1}^{x_n} (s^{(k+1)}(x))^2\, dx > 0.$$

Now recall the definitions of $s(x)$ and the matrix V and notice that

$$\mathbf{c}' V \mathbf{c} = \sum_{j=1}^{n} c_j s(x_j).$$

The result follows. ∎

LEMMA 2.3. *The matrix $C'V^{-1}C$ is invertible.*

Proof. Suppose that $C'V^{-1}C\mathbf{x} = \mathbf{0}$. Let $\mathbf{c} = V^{-1}C\mathbf{x}$. Then $C'\mathbf{c} = \mathbf{0}$, that is, $\mathbf{c} \in N(C')$ and

$$\begin{aligned} \mathbf{c}'V\mathbf{c} &= \mathbf{x}'C'V^{-1}VV^{-1}C\mathbf{x} \\ &= \mathbf{x}'C'V^{-1}C\mathbf{x} \\ &= \mathbf{x}'\mathbf{0} = 0. \end{aligned}$$

Hence by Lemma 2.2, $\mathbf{c} = \mathbf{0}$ and therefore $C\mathbf{x} = \mathbf{0}$. But C is of full rank and so $\mathbf{x} = \mathbf{0}$. ∎

With P as in (2.0), let Q now represent a projector onto the polynomials of degree k given by

$$Q\mathbf{f} = [1, x, ..., x^k]\, M\mathbf{f},$$

where $M \in \mathbb{R}^{(k+1)\times n}$ is such that $MC = I_{k+1}$.

THEOREM 2.4. *The representation $s(x) = (P \oplus Q)\mathbf{s}$ holds for all* natural *splines, $s(x)$, of degree $(2k+1)$ with knots $x_1 < x_2 < \cdots < x_n$ if and only if $M = (C'V^{-1}C)^{-1}C'V^{-1}$, that is, Q is the least-squares projector with weight $W = V^{-1}$.*

Proof. First suppose that $M=(C^tV^{-1}C)^{-1}C^tV^{-1}$. Consider any natural spline $s(x)$. Let $t(x)=P\oplus Q\mathbf{s}=[|x-x_1|^{2k+1}, ..., |x-x_n|^{2k+1}]\, V^{-1}(I-CM)\mathbf{s}+[1, x, ..., x^k]\, M\mathbf{s}$. By the uniqueness of natural interpolating splines we need only show that $t(x)$ interpolates $s(x)$ and that $t(x)$ is natural. It is clear that $t(x)$ interpolates. To show that $t(x)$ is natural, write

$$t(x)=\sum_{j=1}^{n} c_j\,|x-x_j|^{2k+1}+\sum_{j=0}^{k} a_j x^j,$$

where $\mathbf{c}=V^{-1}(I-CM)\mathbf{s}$ and $\mathbf{a}=M\mathbf{s}$. As was stated in the proof of Lemma 2.2, $t(x)$ is natural if $\mathbf{c}\in N(C^t)$. But

$$\begin{aligned} C^t\mathbf{c} &= C^tV^{-1}(I-CM)\mathbf{s} \\ &= C^tV^{-1}(I-C(C^tV^{-1}C)^{-1}C^tV^{-1})\mathbf{s} \\ &= C^tV^{-1}\mathbf{s}-(C^tV^{-1}C)(C^tV^{-1}C)^{-1}C^tV^{-1}\mathbf{s} \\ &= C^tV^{-1}\mathbf{s}-C^tV^{-1}\mathbf{s}=\mathbf{0}. \end{aligned}$$

Conversely, if $s(x)=P\oplus Q\mathbf{s}$ is natural then again we must have $\mathbf{0}=C^t\mathbf{c}=C^tV^{-1}(I-CM)\mathbf{s}=C^tV^{-1}\mathbf{s}-(C^tV^{-1}C)\,M\mathbf{s}$. Hence, as $C^tV^{-1}C$ is nonsingular,

$$M\mathbf{s}=(C^tV^{-1}C)^{-1}\,C^tV^{-1}\mathbf{s}.$$

Since $\mathbf{s}\in\mathbb{R}^n$ is arbitrary we must have

$$M=(C^tV^{-1}C)^{-1}\,C^tV^{-1}. \quad \blacksquare$$

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